

# Spinor Derivation of Quasilocal Mean Curvature Mass in General Relativity

Stephen C. Anco

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**Abstract** A spinor derivation is presented for quasilocal mean-curvature mass of spacelike 2-surfaces in General Relativity. The derivation is based on the Sen-Witten spinor identity and involves the introduction of novel nonlinear boundary conditions related to the Dirac current of the spinor at the 2-surface and the tangential flux of a boundary Dirac operator, as well the use of a spin basis adapted to the mean curvature frame of the 2-surface normal space. This setting may provide an alternative approach to a positivity proof for mean-curvature mass based on showing that Witten's equation admits a spinor solution satisfying the proposed nonlinear boundary conditions.

## 1 Introduction

In General Relativity there is substantial interest in extending Witten's spinor proof for positivity of the ADM mass [1, 2] in asymptotically flat spacetimes to the setting of a quasilocal mass for spacelike 2-surfaces. Two major difficulties with such a proof have been, firstly, how to relate the 2-surface side of the Sen-Witten spinor identity to a well-defined quasilocal mass or energy expression with satisfactory properties [3]; and secondly, what boundary conditions to impose in Witten's equation on the spinor at the 2-surface. Resolving these obstacles could provide a new proof of positivity for some of the several known quasilocal mass-energy definitions (see [4] for a comprehensive review) or perhaps even suggest a more fully satisfactory definition based directly on spinors.

In this paper, a formal spinor derivation is presented for quasilocal mean-curvature mass and its variants studied in recent work [5–10]. The key ideas here will involve the introduction of nonlinear boundary conditions for Witten's equation that are related to the Dirac current of the spinor at the 2-surface and also to the tangential flux of a Dirac operator on the 2-surface, combined with a spin basis adapted to the mean curvature frame of the 2-surface normal bundle, which will be used for evaluating the Sen-Witten identity in Sect. 2. Remarks toward a positivity argument are made in Sect. 3, by showing that the positivity of the

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S.C. Anco (✉)

Department of Mathematics, Brock University, St Catharines, ON, L2S 3A1 Canada  
e-mail: sanc@brocku.ca

mean-curvature mass reduces to existence of a spinor solution of Witten’s equation subject to the proposed nonlinear boundary conditions on a spacelike hypersurface that spans the 2-surface. This argument extends to a Hamiltonian form of mean-curvature mass discussed in Sect. 4. In addition, the standard treatment of horizons as inner boundaries for Witten’s equation using chiral boundary conditions [11] (at the horizon) is shown to carry through in the present setting for mean-curvature mass in Sect. 5.

By way of concluding remarks in Sect. 6, a link is pointed out between mean-curvature mass and a positive quasilocal spinorial mass defined by introducing a mean curvature variant of chiral boundary conditions (in place of the previous nonlinear boundary conditions) for which the existence and uniqueness of solutions of Witten’s equation can be rigorously established [12].

A 4-spinor formalism [11] will be used throughout, employing Dirac spinors  $\psi$  and orthonormal frames  $e_a$  and coframes  $e^a$ ,  $a = 0, 1, 2, 3$ . Numerical gamma matrices  $\gamma_a$  attached to an orthonormal frame will satisfy the Clifford algebra  $\gamma_a\gamma_b + \gamma_b\gamma_a = 2g_{ab}\mathbb{1}$  in terms of frame components  $g_{ab} = \text{diag}(-1, +1, +1, +1)$  of the spacetime metric tensor  $g = g_{ab}e^a \otimes e^b$ , such that  $\gamma_0$  is anti-hermitian while  $\gamma_i$  is hermitian. With these standard conventions the Dirac current  $\bar{\psi}\gamma(e)\psi \equiv -\xi(\psi)$  of a spinor is a timelike past-pointing vector field such that  $\xi^0(\psi) = |\psi|^2$  is the norm of  $\psi$ , where  $\gamma(e) = \gamma^a e_a$  is the dual to the soldering form. The covariant derivative operator  $\nabla$  acting on spinor fields will be given by  $\nabla = \partial + \frac{1}{4}\gamma_{ab}\Gamma^{ab}$  with  $\gamma_{ab} \equiv \frac{1}{2}[\gamma_a, \gamma_b]$  where  $\partial$  denotes the coordinate derivative operator and  $\Gamma^{ab}$  denotes the spin connection 1-form, given in terms of the coframe by  $\Gamma_a^b = \nabla_a e^b - e_a \lrcorner (de^b)$ . For a spacelike 2-surface  $S$  spanned by a smooth spacelike hypersurface  $\Sigma$  in a spacetime  $(M, g)$ , the orthonormal frame  $e_a$  will be adapted to  $\Sigma$  and  $S$  so that  $e_0$  is normal to  $\Sigma$  and future-pointing,  $e_1|_S$  is orthogonal to  $S$  in  $\Sigma$  and outward directed,  $e_a|_S$  is tangential to  $S$  for  $a = 2, 3$ . The 2-surface metric tensor will be denoted  $\sigma = g|_S = g_{ab}(e^a \otimes e^b)|_S$ . Spacetime coordinates will also be adapted to  $\Sigma$  and  $S$  by use of a coordinate derivative  $\partial_\Sigma$  on  $\Sigma$  and its restriction  $\partial_S$  on  $S$ . (In general subscripts  $S, \Sigma$  will denote a restriction to tangent spaces  $T(S), T(\Sigma)$  or cotangent spaces  $T^*(S), T^*(\Sigma)$ , as appropriate.)

## 2 Mean-Curvature Mass

To proceed, introduce

$$H \equiv \kappa(e_1)e_1 - \kappa(e_0)e_0 \tag{1}$$

which defines the mean curvature vector [10, 13, 14] of  $S$  in  $M$ , where  $\kappa(e_0), \kappa(e_1)$  are the extrinsic scalar curvatures (trace of the second fundamental form) of  $S$  relative to the normal frame  $\{e_0, e_1\}$  of  $S$ , and write [10]

$$H_\perp \equiv \kappa(e_1)e_0 - \kappa(e_0)e_1 = *H \tag{2}$$

for the dual of  $H$  in the normal space of  $S$ . The absolute norm

$$|H| = |H_\perp| \equiv \sqrt{|\kappa(e_1)|^2 - \kappa(e_0)^2} \tag{3}$$

defines the scalar mean curvature of  $S$  in  $M$ . Note the vectors  $H, H_\perp$ , as well as the scalar  $|H|$ , are independent of choice of a normal frame (they display invariance under boosts and reflections of  $e_0, e_1$ ) and hence are geometrically well-defined given just the 2-surface

and its extrinsic geometry in spacetime. These vectors satisfy the convention that if  $M$  is Minkowski space and  $S$  is a convex 2-surface lying in a spacelike hyperplane then  $H$  is spacelike and outward directed, while  $H_{\perp}$  is timelike and future-pointing, and the norm of  $H$  agrees with the Euclidean mean curvature of  $S$ . An important geometric property of  $H_{\perp}$  is that  $S$  has no expansion in this direction in spacetime

$$\kappa(H_{\perp}) = \frac{1}{2} \text{tr}(\mathfrak{L}_{H_{\perp}}\sigma) = 0, \tag{4}$$

namely the extrinsic scalar curvature of  $S$  in the  $H_{\perp}$  direction vanishes, while the extrinsic scalar curvature of  $S$  in the direction orthogonal to  $H_{\perp}$  is given by the norm of  $H$ ,

$$\kappa(H) = \frac{1}{2} \text{tr}(\mathfrak{L}_H\sigma) = H \cdot H. \tag{5}$$

Hereafter  $S$  is assumed to have  $H$  spacelike, so  $H_{\perp}$  is timelike and  $|H| = \sqrt{H \cdot H} = |H_{\perp}| = \sqrt{-H_{\perp} \cdot H_{\perp}}$  is non-negative.

The quasilocal mean-curvature mass of  $S$  is defined by the surface integral

$$E(S; \sigma) \equiv \frac{1}{8\pi} \int_S (|H|_{\text{flat}} - |H|) dS \tag{6}$$

where  $|H|_{\text{flat}}$  is the Euclidean mean curvature given by an isometric embedding of  $(S, \sigma)$  into a spacelike hyperplane in Minkowski space. (By Weyl’s theorem [15], such an embedding exists and is unique up to rigid motions if  $S$  has positive Gaussian curvature.) As shown in recent work [10], this mass has good geometric properties: it agrees with the ADM mass in a large sphere limit at spatial infinity in asymptotically flat spacetimes, and it is bounded below by (twice) the irreducible mass  $M_{\text{irr}} \equiv \sqrt{A/16\pi}$  at apparent horizons (namely, 2-surfaces whose scalar mean curvature vanishes,  $|H| = 0$  on  $S$ ) where  $A$  denotes the area of  $S$ . Most importantly, the mean-curvature mass is non-negative in all spacetimes that satisfy the dominant energy condition, as proved in [9]. This positivity indicates that mean-curvature mass may be very useful in the context of geometric analysis problems in General Relativity. On the other hand, the physical meaning of this mass is not so clear because it fails to vanish [4] for certain 2-surfaces (not lying in hyperplanes) in Minkowski space.

The starting point for a spinor derivation is the Witten-Nester 2-form [1, 2, 16, 17] given by  $\bar{\psi}\gamma_3\gamma(e) \wedge \nabla\psi$ , where  $\gamma(e) = e^a\gamma_a$  is the soldering form, and  $\gamma_3 \equiv \gamma_0\gamma_1\gamma_2\gamma_3$ . Write

$$\nabla_{\Sigma} = \partial_{\Sigma} + \frac{1}{4}\gamma_{ab}\Gamma_{\Sigma}^{ab}, \quad \mathcal{D} = \partial_S + \frac{1}{4}\gamma_{ab}\Gamma_S^{ab} \tag{7}$$

respectively for the spatial covariant derivative and the 2-surface intrinsic covariant derivative acting on spinors; a slash will be used to denote the contraction of these derivative operators with the soldering form  $\gamma(e)$ . Likewise

$$\not{\nabla}_S \equiv \gamma^a\nabla_a = \not{\mathcal{D}} + \frac{1}{2}\kappa(e_0)\gamma^0 + \frac{1}{2}\kappa(e_1)\gamma^1 + \not{\omega}_a(e)\gamma^a\gamma^1\gamma^0 \tag{8}$$

denotes a Dirac operator associated to  $S$  as a 2-surface sitting in  $M$ . Recall,  $\kappa(e_0) \equiv e^a \cdot \nabla_a e_0$  and  $\kappa(e_1) \equiv e^a \cdot \nabla_a e_1$  are the extrinsic scalar curvatures of  $S$  in the normal directions  $\{e_0, e_1\}$ , and  $\not{\omega}_a(e) \equiv e_1 \cdot \nabla_a e_0$  is the twist of the normal frame in the tangential directions  $\{e_a\}$ . Then, for any smooth spinor  $\psi$  and smooth hypersurface  $\Sigma$  with a 2-surface boundary  $S = \partial\Sigma$ ,

integration of the Witten-Nester 2-form over  $S$  followed by use of Stokes’ theorem, along with some gamma matrix algebra, gives the well-known Sen-Witten identity:

$$\int_S (\psi^\dagger \gamma^1 \nabla_S \psi + \text{c.c.}) dS = 2 \int_\Sigma (-\nabla_\Sigma \psi^\dagger \cdot \nabla_\Sigma \psi + T_{0a} \psi^\dagger \gamma^0 \gamma^a \psi + (\nabla_\Sigma \psi)^\dagger (\nabla_\Sigma \psi)) d\Sigma \tag{9}$$

(c.c. stands for the complex conjugate of the preceding term) where

$$8\pi T_{ab} \equiv \text{Ric}(e_a, e_b) - \frac{1}{2} g_{ab} R \tag{10}$$

defines the stress-energy of  $(M, g)$  in terms of the spacetime Ricci curvature tensor, with  $R = g^{ab} \text{Ric}(e_a, e_b)$  being the scalar curvature.

For the sequel it will be natural to introduce the following geometric spinor operators (see [4, 12] for a summary of relevant mathematical background):

$$\mathcal{D}_a \equiv D_a + \frac{1}{2} \gamma^1 \gamma^0 \varpi_a(e) \tag{11}$$

is a linear combination of the spinorial edth operators [11, 18]; and

$$\hat{\mathcal{D}} \equiv \gamma^1 \gamma^a \mathcal{D}_a, \quad \hat{\mathcal{D}} \equiv \gamma^1 \gamma^a \mathcal{D}_a \tag{12}$$

are boundary flux operators related to the Dirac operator in Witten’s equation [11, 19] as seen later. (Note that  $\gamma^1 \gamma^a$  gives an alternate representation of the Clifford algebra over  $T(S)$ , the more obvious representation being  $\gamma^a$  induced from the Clifford algebra over  $T(M)$ .) These operators  $\hat{\mathcal{D}}$  and  $\mathcal{D}$  are frame-invariant and depend just on the intrinsic geometry of  $S$ .

Now let  $\Sigma_\perp$  be a smooth spacelike hypersurface spanning  $S$  such that it is orthogonal to the dual mean curvature vector  $H_\perp$  at  $S$  in  $M$ . The corresponding adapted orthonormal frame  $e_a$  has  $e_0|_S = \hat{H}_\perp \equiv |H|^{-1} H_\perp$  normal to  $\Sigma_\perp$  at  $S$ ,  $e_1|_S = \hat{H} \equiv |H|^{-1} H$  orthogonal to  $S$  in  $\Sigma_\perp$ ,  $e_a|_S$  tangential to  $S$  as before; this frame will be distinguished by placing a hat atop frame-dependent spinor operators, vectors and 1-forms. Let

$$\varpi \equiv \hat{\varpi}(e) = (\hat{H} \cdot \nabla_a \hat{H}_\perp) e^a \tag{13}$$

which denotes the twist covector of the mean curvature frame of  $S$ . In this geometrically preferred frame the covariant edth operator and boundary flux operator are given by

$$\hat{\mathcal{D}} = \mathcal{D} + \frac{1}{2} \gamma^1 \gamma^0 \varpi, \quad \hat{\mathcal{D}} = \gamma^1 \gamma^a \hat{\mathcal{D}}_a, \tag{14}$$

while the boundary Dirac operator takes the form

$$\nabla_S = \gamma^1 \left( \hat{\mathcal{D}} + \frac{1}{2} |H| \right) \tag{15}$$

due to the property (4) of the mean curvature frame. In particular,  $\hat{\mathcal{D}}$ ,  $\hat{\mathcal{D}}$ , and  $\varpi, H, H_\perp$  all are geometrically well-defined given just the 2-surface  $S$  and its extrinsic geometry in spacetime. The Sen-Witten identity now becomes

$$\int_S (\psi^\dagger \hat{\mathcal{D}} \psi + \text{c.c.} + |H| |\psi|^2) dS = -2 \int_{\Sigma_\perp} (|\nabla_\Sigma \psi|^2 - \xi(\psi) \cdot T_0 - |\nabla_\Sigma \psi|^2) d\Sigma \tag{16}$$

with  $|\nabla_\Sigma \psi| \equiv \sqrt{\nabla_\Sigma \psi^\dagger \cdot \nabla_\Sigma \psi}$  and

$$T_0 \equiv T_0^a e_a, \quad \xi(\psi) \equiv -\bar{\psi} \gamma^a e_a \psi, \tag{17}$$

as defined in the adapted orthonormal frame.

The aim is now to relate the surface integral terms in the spinor identity (16) to a quasi-local mass expression. This will be accomplished through considering the tangential Dirac current

$$\xi_{||}(\psi_S) \equiv -(\bar{\psi} \gamma^a e_a \psi)|_S \tag{18}$$

and the tangential flux

$$\mathcal{F}(\psi_S) \equiv (\psi^\dagger \mathcal{D} \psi)|_S + \text{c.c.} \tag{19}$$

of  $\psi_S \equiv \psi|_S$  connected with an isometric embedding of  $S$  into a spacelike hyperplane in Minkowski space. Note  $(\nabla H_\perp)_{\text{flat}} = 0$  and hence  $(\varpi)_{\text{flat}} = 0$  holds on the embedded 2-surface  $S_{\text{flat}}$ , so thus  $\mathcal{D}_{\text{flat}} = \mathcal{D}$  and  $(\nabla_S)_{\text{Mink}} = \mathcal{D} + \frac{1}{2}|H|_{\text{flat}}\gamma^1$  using an orthonormal frame  $(e_a)_{\text{Mink}}$  adapted to the hyperplane and the embedded 2-surface (whose mean curvature frame  $\{\hat{H}_\perp, \hat{H}\}_{\text{flat}}$  then coincides with the adapted normal frame for  $a = 0, 1$  and whose tangential frame pulls back to the one on  $S$  for  $a = 2, 3$ ). With respect to this embedding, choose  $\psi$  to be a parallel (covariantly constant) spinor in Minkowski space,  $(\nabla \psi)_{\text{Mink}} = 0$ , with

$$\xi^a(\psi) = (|\psi|^2, 0, 0, 0), \quad |\psi| = \text{const.} \tag{20}$$

More geometrically, the embedded Dirac current vector of  $\psi$  at  $S_{\text{flat}}$  is aligned with the dual mean curvature vector of  $S_{\text{flat}}$ ,

$$(\xi(\psi_S) \wedge H_\perp)_{\text{flat}} = 0, \tag{21}$$

and has constant absolute norm,

$$|\xi(\psi_S)| = |\psi|^2 = \text{const.} \tag{22}$$

Moreover,  $\psi$  satisfies the boundary Witten equation

$$(\nabla_S \psi_S)_{\text{Mink}} = \gamma^1 \left( \mathcal{D} \psi_S + \frac{1}{2}|H|_{\text{flat}} \psi_S \right) = 0 \tag{23}$$

on the embedded 2-surface  $S_{\text{flat}}$ , and so the tangential flux

$$\mathcal{F}(\psi_S) = -|H|_{\text{flat}} |\psi_S|^2 \tag{24}$$

is a constant multiple of the Euclidean mean curvature of  $S_{\text{flat}}$ . Note, for comparison, the boundary Witten equation on  $S$  itself in the spacetime  $M$  would look like

$$\begin{aligned} \nabla_S \phi &= \mathcal{D} \phi + \frac{1}{2} \kappa(e_1) \gamma^1 \phi + \frac{1}{2} \kappa(e_0) \gamma^0 \phi + \varpi_\perp(e) \gamma^a \gamma^1 \gamma^0 \phi \\ &= \gamma^1 \left( \mathcal{D} \phi + \frac{1}{2} \kappa(e_1) \phi + \frac{1}{2} \kappa(e_0) \gamma^1 \gamma^0 \phi \right) = 0 \end{aligned} \tag{25}$$

relative to a general orthonormal frame  $e_a$ , for any spinor  $\phi$ . Now, substitution of (22) and (24) into the Sen-Witten identity (16) yields a main result:

$$\int_S \psi^\dagger \widehat{\mathcal{D}} \psi + \text{c.c.} + |H| |\psi|^2 dS = \int_S (\mathcal{F}(\psi_S) + \varpi \cdot \xi_{\parallel}(\psi_S) + |H| |\xi(\psi_S)|) dS$$

$$= -|\psi_S|^2 \int_S (|H|_{\text{flat}} - |H|) dS, \quad |\psi_S| = \text{const.} \quad (26)$$

Note there is no loss of generality in scaling  $\psi$  by a constant factor so that  $|\psi_S| = 1$ .

**Theorem 1** *Let  $S$  be a spacelike 2-surface whose mean curvature vector  $H$  is spacelike and whose Gaussian curvature is positive. Fix a Dirac spinor  $\psi$  in Minkowski space satisfying boundary conditions  $\mathcal{F}(\psi_S) = -|H|_{\text{flat}}, \xi_{\parallel}(\psi_S) = 0$ , and  $|\psi_S| = 1$  on  $S$  embedded into a spacelike hyperplane. (In particular such boundary conditions hold when  $\psi$  is any parallel spinor,  $(\nabla \psi)_{\text{flat}} = 0$ , aligned and normalized so that  $\xi(\psi_S)_{\text{flat}} = (\hat{H}_{\perp})_{\text{flat}}$  holds in the embedding.) Then the surface integral terms in the Sen-Witten identity are a multiple of the mean-curvature mass (6) of  $S$ :*

$$8\pi E(S; \sigma) = - \int_S \psi^\dagger \widehat{\mathcal{D}} \psi + \text{c.c.} + |H| |\psi|^2 dS. \quad (27)$$

### 3 Remarks on Positivity

This spinor derivation of the mean curvature mass allows the possibility of modifying Witten’s positivity argument as follows. Impose on  $\psi$  the Witten equation

$$\nabla_{\Sigma} \psi = 0 \quad (28)$$

subject to the nonlinear boundary conditions

$$\mathcal{F}(\psi_S) = \mathcal{F}(\phi) = -|H|_{\text{flat}}, \quad (29)$$

$$\xi_{\parallel}(\psi_S) = \xi_{\parallel}(\phi) = 0, \quad (30)$$

$$|\psi_S| = |\phi| = 1, \quad (31)$$

under the isometric embedding of  $(S, \sigma)$  into a spacelike hyperplane in Minkowski space, where  $\phi$  is the restriction to  $S$  of a parallel spinor whose Dirac current vector is aligned with the dual mean curvature vector of  $S$  in the embedding,  $\hat{\xi}(\phi)_{\text{flat}} = (\hat{H}_{\perp})_{\text{flat}}$ . Note  $\phi$  then satisfies the boundary Witten equation (in Minkowski space)

$$(\nabla_S \phi)_{\text{Mink}} = \gamma^1 \left( \mathcal{D} \phi + \frac{1}{2} |H|_{\text{flat}} \phi \right) = 0 \quad (32)$$

such that the boundary Dirac current vector has unit absolute norm

$$|\xi(\phi)| = |\phi|^2 = 1. \quad (33)$$

These boundary conditions specify the tangential Dirac current of  $\psi$  on  $S$  and the tangential flux of  $\psi$  on  $S$ , along with the norm of  $\psi$  (which is given by the projection of the Dirac current vector of  $\psi$  in the timelike mean curvature direction  $H_{\perp}$  at  $S$ ,  $\xi(\psi_S) \cdot \hat{H}_{\perp} = -|\psi_S|^2$ ).

The boundary value problem given by (28–31) on  $\psi$  constitutes a first-order elliptic PDE system with the right number of boundary conditions (half of the degrees of freedom of the spinor). To give a more rigorous indication of well-posedness, it would be natural to first study the linearized equations as obtained in a weak gravitational limit (where the spacetime metric is a perturbation of the Minkowski metric). This would yield a linear, non-homogeneous boundary value problem in Minkowski space to which standard Green’s function techniques could be applied to verify if there exist solutions  $\psi$ . Such an analysis will be left for investigation elsewhere, and the rest of the argument will now be formal.

From Witten’s equation (28) in the nonlinear boundary value problem, since the hypersurface  $\Sigma_\perp$  is orthogonal to the timelike mean curvature normal of  $S$ , the normal covariant derivative of  $\psi$  at  $S$  in  $\Sigma_\perp$  is given by

$$(\nabla_1 \psi)|_S = -\gamma^1 \nabla_S \psi_S = -\left(\widehat{\mathcal{R}}\psi_S + \frac{1}{2}|H|\psi_S\right) \tag{34}$$

and consequently the normal flux of  $\psi$  on  $S$  is precisely the same as the Sen-Witten surface integral density expression. In particular, under the boundary conditions (29–31),

$$(\psi^\dagger \nabla_1 \psi)|_S + \text{c.c.} = -(\psi_S^\dagger \widehat{\mathcal{R}}\psi_S + \text{c.c.} + |H||\psi_S|^2) = |H|_{\text{flat}} - |H| \tag{35}$$

so thus the normal flux of  $\psi$  on  $S$  is equal to the difference of the scalar mean curvature of  $S$  (as a 2-surface in  $M$ ) and the Euclidean mean curvature of  $S$  (as an embedded 2-surface in Minkowski space).

As a result the spinor derivation of mean-curvature mass (27) holds (similarly to Theorem 1) and is directly related to the normal flux of the spinor in Witten’s equation at the 2-surface in spacetime. Then (16) and (28) yield

$$8\pi E(S; \sigma) = 2 \int_{\Sigma_\perp} (|\nabla_\Sigma \psi|^2 - \xi(\psi) \cdot T_0) d\Sigma \geq 0 \tag{36}$$

since

$$-\xi(\psi) \cdot T_0 \geq 0 \tag{37}$$

if the local energy-momentum vector field  $T_0$  associated with  $\Sigma_\perp$  is timelike and future-pointing (which is just the dominant energy condition [20, 21] on the spacetime stress-energy tensor  $T_{ab}$ ).

**Proposition 2** *Let  $S$  be a 2-surface as in Theorem 1. If a smooth solution  $\psi$  of the Witten equation (28) exists satisfying the nonlinear boundary conditions (29–31) on a hypersurface  $\Sigma_\perp$  spanning  $S$  then the mean-curvature mass (6) of  $S$  is non-negative provided the spacetime  $(M, g)$  obeys the dominant energy condition.*

### 4 Twist-Free 2-Surfaces and Mean-Curvature Mass

A 2-surface  $S$  is said to be *convex* if its mean curvature vector  $H$  is spacelike, or equivalently the dual mean curvature vector  $H_\perp$  is timelike; and said to be *twist-free* if the twist of these normal vectors  $\varpi_\perp(H) = H \cdot \nabla_\perp H_\perp$  vanishes, so therefore a convex twist-free  $S$  possesses a mean curvature normal frame  $\{\widehat{H}_\perp, \widehat{H}\}$  whose twist is zero,  $\varpi = 0$ .

For any such 2-surface, an elegant version of the spinor derivation can be formulated using  $SU(2)$  spinors on a maximal spacelike hypersurface  $\Sigma$ . Firstly, recall that the Witten equation on a maximal hypersurface reduces to the 3-dimensional Witten equation by means of the spatial Dirac operator decomposition

$$\nabla_{\Sigma} = \not{D} + \frac{1}{2}\gamma^0 \text{tr}(K(e_0)) \tag{38}$$

with  $D = \partial_{\Sigma} + \frac{1}{4}\gamma_{ij}\Gamma_{\Sigma}^{ij}$  being the hypersurface covariant derivative, where  $h = g|_S = g_{ij}(e^i \otimes e^j)|_{\Sigma}$  is the spatial metric tensor in terms of the spatial coframe  $e^i$ ,  $i = 1, 2, 3$ , intrinsic to  $(\Sigma, h)$ , and where  $K(e_0) = \frac{1}{2}\mathcal{L}_{e_0}h$  is the second fundamental form of  $\Sigma$  in terms of the spatial metric  $h$ . Here  $\not{D}$  is the spatial (3-dimensional) Dirac operator associated to  $\Sigma$ . Secondly, recall that  $SU(2)$  spinors,  $\psi_{\Sigma}$ , on a spacelike hypersurface are defined via projection operators  $\mathcal{P}_{\pm}^0 = \frac{1}{2}(\mathbb{1} \pm i\gamma^0)$  by the conditions  $\mathcal{P}_{+}^0\psi = \psi$  or equivalently  $\mathcal{P}_{-}^0\psi = 0$  applied to a Dirac spinor  $\psi$  [21, 22]. This projection reduces the number of linearly independent real components of  $\psi_{\Sigma}$  to four. A key property of these spinors is their compatibility with the spatial Dirac operator,

$$\not{D}\mathcal{P}_{\pm}^0 = \mathcal{P}_{\mp}^0\not{D}, \tag{39}$$

as seen by using  $[\mathcal{P}_{\pm}^0, \gamma^0] = 0$  and  $= \mathcal{P}_{\pm}^0\gamma^i - \gamma^i\mathcal{P}_{\mp}^0 = 0$ .

To set up the derivation of the mean-curvature mass, assume  $S$  is spanned by a maximal spacelike hypersurface  $\Sigma_{\perp}$  orthogonal to  $H_{\perp}$  at  $S$ . Suppose there exists an  $SU(2)$  spinor  $\psi_{\Sigma}$  satisfying the spatial Witten equation

$$\not{D}\psi_{\Sigma} = 0 \tag{40}$$

and nonlinear boundary conditions (29) and (31) under an isometric embedding of  $(S, \sigma)$  into a spacelike hyperplane in Minkowski space, where  $\psi_S$  is the restriction of  $\psi_{\Sigma}$  to  $S$ . Here  $\phi$  is a parallel  $SU(2)$  spinor in Minkowski space,  $(\nabla\phi)_{\text{Mink}} = 0$ , with its Dirac current vector aligned with the dual mean curvature vector of  $S$  in the embedding,  $\xi(\phi)_{\text{flat}} = (\hat{H}_{\perp})_{\text{flat}}$ . As before,  $\phi$  thus satisfies the embedded boundary Witten equation (32) with the boundary Dirac current obeying the normalization (33).

The elegance of  $SU(2)$  spinors now comes into play in linking the Sen-Witten identity to the mean-curvature mass in a Hamiltonian form [10] as follows. First, using the projection operators it can be easily shown that the Dirac current of  $\psi_{\Sigma}$  is orthogonal to  $\Sigma_{\perp}$ ,

$$\xi(\psi_{\Sigma}) = |\psi_{\Sigma}|^2 e_0. \tag{41}$$

This identity gives a simple spinor parametrization of the timelike mean curvature vector of  $S$  since  $e_0|_S = \hat{H}_{\perp}$ .

**Theorem 3** *On a convex twist-free spacelike 2-surface  $S = \partial\Sigma_{\perp}$ , the surface integral terms in the Sen-Witten identity on  $\psi_{\Sigma}$  subject to boundary conditions (29) and (31) in the adapted mean curvature frame are given by*

$$-\int_S (\psi_{\Sigma}^{\dagger} \widehat{\not{D}} \psi_{\Sigma} + \text{c.c.} + |H||\psi_{\Sigma}|^2) dS = \int_S (|H|_{\text{flat}} - |H|) dS = \int_S \xi \cdot P dS - \int_S (\xi \cdot P)_{\text{flat}} dS \tag{42}$$



where  $P \equiv H_{\perp} + \varpi$  is the symplectic vector [23, 24] in a Hamiltonian formulation of the Einstein gravitational field equations using a geometric time flow vector field [10] which is given at  $S$  by the timelike mean curvature vector  $\xi \equiv \hat{\xi}(\psi_S) = \hat{H}_{\perp}$ .

A positivity analysis is now possible from considering (16), (40) and (42), together with the fact that  $\mathcal{D} = \mathcal{V}_{\Sigma}$  since  $\Sigma_{\perp}$  is a maximal hypersurface. This yields the inequality

$$8\pi E(S; \sigma) = 2 \int_{\Sigma_{\perp}} (|\nabla_{\Sigma} \psi_{\Sigma}|^2 + T_{00} |\psi_{\Sigma}|^2) d\Sigma \geq 0 \tag{43}$$

whenever  $T_{00} \geq 0$  (which is the weak energy condition [20, 21] on the stress-energy tensor  $T_{ab}$ ).

**Proposition 4** *Let  $S$  be a convex twist-free spacelike 2-surface with positive Gaussian curvature, spanned by an adapted maximal spacelike hypersurface  $\Sigma_{\perp}$  in spacetime  $(M, g)$ . If the spatial Dirac boundary value problem (40), (29) and (31) possesses a smooth  $SU(2)$  spinor solution  $\psi_{\Sigma}$ , then the mean-curvature mass (6) is non-negative provided the weak energy condition holds on  $(M, g)$ .*

Compared with Proposition 2, here a weaker energy condition is sufficient for positivity, since the Dirac current (41) for  $SU(2)$  spinors is hypersurface orthogonal, but there is a stronger restriction on the 2-surface  $S$ .

### 5 Horizons and Chiral Boundary Conditions

In the positivity analysis, if a hypersurface  $\Sigma$  contains an apparent horizon inside  $S$  then the horizon 2-surface should be treated as an inner boundary  $(\partial \Sigma)_{\text{hor}}$ , which is characterized by the vanishing of its scalar mean curvature,  $|H|_{\text{hor}} = 0$ . On such hypersurfaces, the boundary value problem (28–31) needs to be supplemented by imposing the standard horizon boundary condition [11]  $\gamma^1 \gamma^0 \psi|_{\text{hor}} = \psi|_{\text{hor}}$  on the Dirac spinor in Witten’s equation, in terms of an orthonormal frame adapted to  $(\partial \Sigma)_{\text{hor}}$ .

This boundary condition arises from the introduction of chiral projection operators

$$\mathcal{P}_{\pm} = \frac{1}{2} (\mathbb{1} \pm \gamma^1 \gamma^0) \tag{44}$$

characterized by the properties  $[\mathcal{P}_{\pm}, \gamma^a] = 0$ ,  $\mathcal{P}_{\pm} \gamma^1 = \gamma^1 \mathcal{P}_{\mp}$ ,  $\mathcal{P}_{\pm} \gamma^0 = \gamma^0 \mathcal{P}_{\mp}$ , in addition to  $\mathcal{P}_{\pm}^2 = \mathcal{P}_{\pm}$ ,  $\mathcal{P}_{+} \mathcal{P}_{-} = 0$ , and  $\mathcal{P}_{\pm}^{\dagger} = \mathcal{P}_{\pm}$ . Relative to the decomposition  $\psi = \psi_{+} + \psi_{-}$  given by  $\psi_{\pm} \equiv \mathcal{P}_{\pm} \psi$ , the surface integral terms in the Sen-Witten identity look like

$$\int_S 2\psi_{+}^{\dagger} \mathcal{D} \psi_{-} + \text{c.c.} + |H| (|\psi_{+}|^2 + |\psi_{-}|^2) dS \tag{45}$$

after integration by parts with respect to  $\mathcal{D}$ , discarding a total divergence on  $S$  (by Stokes’ theorem). Hence at a horizon  $S = (\partial \Sigma)_{\text{hor}}$ , the flux terms vanish under the boundary condition  $\psi_{-}|_{\text{hor}} = 0$ , while the mean curvature terms are zero due to  $|H|_{\text{hor}} = 0$ , as first shown in [11]. Consequently, the horizon boundary condition implies that the inner boundary makes no contribution to the quasilocal mass so that the positivity analysis leading to Propositions 2 and 4 goes through as before.

### 6 Some Concluding Remarks on Quasilocal Spinorial Mass

It is natural to explore a spinor derivation and positivity analysis of mean-curvature mass based on a chiral boundary condition in place of the nonlinear boundary conditions investigated so far. Let  $\hat{\mathcal{P}}_{\pm}$  denote the chiral projection operators adapted to the mean curvature frame of a 2-surface  $S$  and consider

$$\psi_-|_S = \phi_-|_S \tag{46}$$

where, in a Euclidean embedding of  $(S, \sigma)$  into a spacelike hyperplane in Minkowski space,  $\phi$  is a parallel Dirac spinor whose Dirac current is aligned with the timelike mean curvature vector of  $S$ ,

$$\bar{\xi}(\phi) = (\hat{H}_{\perp})_{\text{flat}}. \tag{47}$$

Note  $[\hat{\mathcal{P}}_{\pm}, \mathcal{D}] = 0$  shows that  $\phi_{\pm}$  obeys the boundary Witten equation

$$\hat{\mathcal{P}}_{\mp}(\nabla_S \phi)_{\text{Mink}} = \gamma^1 \left( \mathcal{D}\phi_{\mp} + \frac{1}{2}|H|_{\text{flat}}\phi_{\pm} \right) = 0. \tag{48}$$

Also note, as a consequence of the alignment (47), that

$$|\phi_+| = |\phi_-| = \frac{1}{\sqrt{2}} \tag{49}$$

since  $0 = \xi^1(\phi) = (\hat{H})_{\text{flat}} \cdot \xi(\phi) = -\phi^{\dagger}\gamma^0\gamma^1\phi = \phi^{\dagger}(\hat{\mathcal{P}}_+ - \hat{\mathcal{P}}_-)\phi = |\phi_+|^2 - |\phi_-|^2$  while  $1 = \xi^0(\phi)_{\text{flat}} = |\phi|^2 = |\phi_+|^2 + |\phi_-|^2$ .

Witten’s equation (28) with this chiral boundary condition (46) is an elliptic boundary value problem for the Dirac spinor

$$\chi = \psi - \phi, \quad \chi_-|_S = 0, \tag{50}$$

satisfying the equation

$$\nabla_{\Sigma}\chi = -\nabla_{\Sigma}\phi_- = -\Gamma\phi \tag{51}$$

where  $\Gamma$  is a certain gamma matrix operator that vanishes in Minkowski space,  $(\Gamma)_{\text{Mink}} = 0$ . Existence and uniqueness of solutions  $\chi$  is rigorously established by the general results stated in [12] for boundary value problems of this kind. It is important to note that, in particular, there are no zero modes for the boundary value problem (50, 51), since if  $\chi_0$  satisfies  $\nabla_{\Sigma}\chi_0 = 0$  on a hypersurface  $\Sigma$  with  $\chi_-|_S = 0$  at a boundary  $S = \partial\Sigma$ , then the Sen-Witten identity evaluated for  $\chi_0$  yields  $\int_{\Sigma} |\nabla_{\Sigma}\chi_0|^2 d\Sigma = \int_{\Sigma} \bar{\xi}(\chi_0) \cdot T_0 d\Sigma \leq 0$  provided the dominant energy condition holds on  $\Sigma$ . This inequality implies  $\nabla_{\Sigma}\chi_0 = 0$  so then  $\chi_0|_{\Sigma}$  must be a spatially parallel spinor which vanishes due to the chiral boundary condition.

To proceed, let  $\psi$  be the unique solution of the boundary value problem (50, 51), and consider the surface integral terms in the Sen-Witten identity (16). Use of the boundary Witten equation (48) for  $\phi_-$  gives  $(\psi_+^{\dagger}\mathcal{D}\psi_-)|_S = (\psi_+^{\dagger}\mathcal{D}\phi_-)|_S = -\frac{1}{2}|H|_{\text{flat}}(\psi_+^{\dagger}\phi_+)|_S$  for the flux terms. The complete Sen-Witten identity then yields the expression

$$\begin{aligned} 8\pi \tilde{E}(S; \sigma, \psi) \equiv & \int_S ((\psi_+^{\dagger}\phi_+ + \text{c.c.})|H|_{\text{flat}} - (|\psi_+|^2 + |\phi_+|^2)|H| \\ & + (\psi_+^{\dagger}\gamma^0\gamma^a\phi_- + \text{c.c.})\varpi_a) dS \geq 0 \end{aligned} \tag{52}$$

assuming the dominant energy condition holds on  $\Sigma$ . The surface integral (52) defines a purely spinorial quasilocal mass  $\tilde{E}(S; \sigma, \psi)$  which can be naturally viewed as a chiral variant of the mean-curvature mass of  $S$ . Interestingly, this variant expression is rigorously positive and should have some good properties as a quasilocal mass. In any case, its positivity may perhaps provide a link to the known positivity [9] of mean-curvature mass.

As it stands, however this positivity property (52) gives only a lower bound on the mean-curvature mass

$$4\pi E(S; \sigma) = \frac{1}{2} \int_S (|H|_{\text{flat}} - |H|) dS \geq - \int_S |\psi_+|^2 (|H|_{\text{flat}} - |H|) + |\psi_+| |\varpi| dS \quad (53)$$

which follows from the elementary estimates  $\psi_+^\dagger \phi_+ + \text{c.c.} \leq |\psi_+|^2 + |\phi_+|^2$  and  $\psi_+^\dagger \gamma^0 \gamma^a \phi_- \varpi_a \leq |\psi_+| |\phi_-| |\varpi|$ , combined with the normalizations (49). A slightly more satisfactory result can be obtained if  $S$  is restricted to be twist-free, in which case the twist term will drop out of the inequality (52).

**Lemma 5** *Suppose  $S$  is twist-free in spacetime and let  $\psi_+|_S$  be the boundary value of the solution of Witten’s equation (28) with the mean curvature chiral boundary condition (46), under a Euclidean embedding of  $S$ . Then the mean curvature of  $S$  obeys the inequality*

$$\int_S \left( \frac{1}{2} + |\psi_+|^2 \right) (|H|_{\text{flat}} - |H|) dS = 8\pi \tilde{E}(S; \sigma, \psi) \geq 0. \quad (54)$$

This inequality would directly imply positivity of the mean-curvature mass  $E(S; \sigma)$  for twist-free 2-surfaces  $S$  if the boundary value  $\psi_+|_S$  has constant norm on  $S$ . A similar conclusion would follow from the more general inequality (52) under stronger conditions on  $\psi_+|_S$  (namely, if  $\psi_+|_S$  is equal to  $\phi_+|_S$  so then  $(\psi_+ \gamma^0 \gamma^a \phi_-)|_S = \xi^a(\phi_-)$  and  $(\psi_+^\dagger \phi_+)|_S = |\phi_+|^2 = |\phi_-|^2$ , implying the twist terms would vanish due to  $(\psi_+ \gamma^0 \gamma^a \phi_- + \text{c.c.})|_S \varpi_a = \xi_{||}(\phi) \cdot \varpi = 0$  while the mean curvature terms would simplify to  $(|\psi_+|^2 + |\phi_+|^2)|_S (|H|_{\text{flat}} - |H|) = |H|_{\text{flat}} - |H|$ ). Clearly, such lines of argument will require a detailed analytical investigation of the mean curvature chiral boundary value problem for Witten’s equation.

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